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# Bifurcation of limit cycles and integrability of planar dynamical systems in complex form

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**Abstract.** We describe a new algorithm for the calculation of Liapunov quantities for two-dimensional systems in terms of the complex form of the system. It is more transparent conceptually and more efficient computationally. We demonstrate the advantages of this approach by deriving the integrability conditions and information about the bifurcation of limit cycles for some cubic systems.

## 1. Introduction

The use of computer algebra has led to significant progress in the investigation of the properties of planar dynamical systems. The calculations involved in, for instance, the bifurcation of limit cycles and the derivation of centre conditions are extremely heavy and are quite impossible to accomplish by hand except in the simplest cases. The expressions arising can be very large, and the difficulties are exacerbated by intermediate expression swell. Inevitably systems are considered in which the limits of available computing capacity are reached. At that stage every effort to reduce the burden of the computations in terms of computing time, but more often of space, is rewarded. Considerable effort has consequently been devoted to improving the algorithms used, thereby extending the range of their applicability. In this paper we show that working with the complex form of the system—that is, with the single differential equation for $z = x + \mathrm{i}y$—does indeed lead to perceptible benefits in some situations. Moreover, the complex form is especially convenient for theoretical work. For example, the conditions under which a system is symmetric in a line can be readily obtained; we have exploited this in several instances. This form has also enabled us to resolve an outstanding question relating to the Kukles system [8]. We conjectured that for this particular cubic system there are no integrability conditions which are not 'persistent'. We are able to prove that this is the case for any system of this type whatever its degree. The approach extends the selection of techniques available for investigating integrability and the bifurcation of limit cycles in dynamical systems. No single approach is likely to be the best choice universally.

Suppose that $P$ and $Q$ are analytic and that $x = y = 0$ is a critical point of focus type of the system

$$\dot{x} = P(x, y) \qquad \dot{y} = Q(x, y). \tag{1.1}$$

We are concerned with two of the questions that should be addressed in order to understand the behaviour of such systems. One is the number of limit cycles which can bifurcate out

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of a critical point and the other is concerned with the conditions under which a critical point is a centre. These are closely related issues, both of which have attracted much attention over the years. For example, a knowledge of the integrability conditions is required in order to determine the number of limit cycles which can bifurcate. Systems of the form (1.1) are used to model various phenomena—an interesting recent example is concerned with the crystallization of agates [14, 16].

We first describe the background and define terms. We write system (1.1) as

$$\begin{aligned} \dot{x} &= \lambda x + y + p_2(x, y) + \cdots + p_k(x, y) + \cdots \\ \dot{y} &= -x + \lambda y + q_2(x, y) + \cdots + q_k(x, y) + \cdots \end{aligned} \tag{1.2}$$

where $p_k$ and $q_k$ are homogeneous polynomials of degree $k$. It is well known that there is an analytic function $V$ such that $\dot{V}$, its rate of change along orbits, is of the form $\Sigma\eta_{2k}(x^2 + y^2)^k$. The $\eta_{2k}$ are the *focal values*, and they are polynomials in the coefficients arising in $P$ and $Q$. The origin is said to be a *fine focus of order* $k$ if $\eta_{2i} = 0$ for $i \leqslant k$ but $\eta_{2k+2} \neq 0$. At most $k$ limit cycles can bifurcate from a fine focus of order $k$ under perturbation of the coefficients in $P$ and $Q$. The stability of the origin is determined by the sign of the first non-vanishing focal value, and it is a centre if all the focal values are zero. A critical point is, of course, a *centre* when all orbits in its neighbourhood are closed. Since system (1.2) is analytic and the linear terms are present there is an analytic first integral whenever the origin is a centre. We describe the system as integrable precisely when the origin is a centre.

In Hilbert's 16th problem questions are raised about the number and possible configurations of limit cycles of (1.1) when $P$ and $Q$ are polynomials. Consider a particular class of such systems. By the Hilbert basis theorem the ideal generated by the focal values in the ring of polynomials in the coefficients occurring in $P$ and $Q$ has a finite basis. Thus for polynomial systems there is $M$ such that $\eta_{2k} = 0$ for $k > M$ if $\eta_{2k} = 0$ for $k \leqslant M$. Thus $M - 1$ is the maximum possible order of the fine focus at the origin, and the critical point is a centre if and only if $\eta_{2k} = 0$ for $k \leqslant M$. However the value of $M$ is not known in advance. In order to find it, and at the same time obtain necessary conditions for integrability, focal values are computed. The coefficients are eliminated by making substitutions from the relations $\eta_2 = \eta_4 = \cdots = \eta_{2\ell-2} = 0$ in $\eta_{2\ell}$ for each $\ell$. The expressions so obtained are called the *Liapunov quantities* and are denoted by $L(k)$, $k = 0, 1, \ldots$. The set of common zeros of the Liapunov quantities coincides with the zero set of the focal values and is the *centre variety*. This reduction process leads to *necessary* conditions for a centre. It can be very demanding computationally, as was seen in [8]; hence anything which leads to less complicated calculations is helpful. The *sufficiency* of the conditions obtained is proved independently using a number of different methods. One of the most useful is described in [12]; it relies on a partly automated systematic search for invariant functions which are used to construct integrating factors.

The focal values may be defined in different ways, for example by requiring that $\dot{V}$ is of the form $\Sigma\eta'_{2k}y^{2k}$, and there are other methods of calculating them. It was proved by Shi [15] that the ideal generated by the focal values is independent of the way in which they are defined and found. The focal values can be calculated by hand only in simple cases. A number of algorithms have been developed to compute them automatically using various computer algebra packages, see, for example, [1–4, 13]. Our procedure, using REDUCE, is described in [6]; we called the algorithm FINDETA. We have used it extensively to obtain information about the number of bifurcating limit cycles and to derive necessary conditions for the origin to be a centre. It has proved efficient and easy to use. The results obtained are described in a number of papers such as [7–11]. In this paper we present another approach to the computation of focal values which, certainly in some situations, is more effective computationally than the methods previously used.

Research on polynomial systems has progressed by consideration of particular classes of systems. Thus quadratic systems were the first to be studied intensively. They are by now reasonably well understood, though the maximum number of limit cycles is still not known. Liénard systems (of various degrees) provide another example of a class which has been extensively investigated. More recently much effort has been devoted to cubic systems and the examples contained in this paper are of this kind.

It is sometimes helpful to write a polynomial system of the form (1.2) in terms of the complex variable $z = x + \mathrm{i}y$. When the origin is a fine focus $\eta_2 = \lambda = 0$ and (1.2) becomes

$$\mathrm{i}\dot{z} = z + \sum_{\substack{i+j\geqslant 2\\ i,j\geqslant 0}} A_{ij} z^i \bar{z}^j. \tag{1.3}$$

This is the form which we use in the new algorithm described in section 2.

## 2. The focal values

We consider the system

$$\mathrm{i}\dot{z} = z + f_2 + f_3 + \cdots$$

where $f_k$ is a homogeneous polynomial in $z$ and $\bar{z}$ of degree $k$, and write

$$f_k = \sum_{\ell=0}^{k} A_{k-\ell,\ell} z^{k-\ell} \bar{z}^{\ell}.$$

We seek a function $V = V_2 + V_3 + V_4 + \cdots$, where $V_k$ is a homogeneous polynomial in $z, \bar{z}$ of degree $k$ with $V_2 = z\bar{z}$, such that

$$\dot{V} = \eta_4 (z\bar{z})^2 + \eta_6 (z\bar{z})^3 + \cdots.$$

Now

$$\dot{V} = \frac{1}{\mathrm{i}}\frac{\partial V}{\partial z}(z + f_2 + f_3 + \cdots) - \frac{1}{\mathrm{i}}\frac{\partial V}{\partial \bar{z}}(\bar{z} + \bar{f}_2 + \bar{f}_3 + \cdots).$$

Therefore

$$\mathrm{i}\dot{V} = (\bar{z} + V_{3,z} + V_{4,z}\cdots)(z + f_2 + \cdots) - (z + V_{3,\bar{z}} + V_{4,\bar{z}}\cdots)(\bar{z} + \bar{f}_2 + \cdots) \tag{2.1}$$

where $V_{k,z}$ and $V_{k,\bar{z}}$ denote $\partial V_k/\partial z$ and $\partial V_k/\partial \bar{z}$ respectively. Denote the terms on the right-hand side of (2.1) of degree $k$ by $R_k$. Then

$$R_k = zV_{k,z} - \bar{z}V_{k,\bar{z}} + \sum_{\ell=2}^{k-1} f_{k-\ell+1} V_{\ell,z} - \sum_{\ell=2}^{k-1} \bar{f}_{k-\ell+1} V_{\ell,\bar{z}}. \tag{2.2}$$

For $k \geqslant 3$ write

$$V_k = \sum_{j=0}^{k} V_{k-j,j} z^{k-j} \bar{z}^j.$$

The first two terms in the right-hand side of (2.2) are

$$\begin{aligned} zV_{k,z} - \bar{z}V_{k,\bar{z}} &= z\sum_{j=0}^{k}(k-j)V_{k-j,j} z^{k-j-1}\bar{z}^j - \bar{z}\sum_{j=0}^{k} jV_{k-j,j} z^{k-j}\bar{z}^{j-1} \\ &= \sum_{j=0}^{k}(k-2j) z^{k-j}\bar{z}^j V_{k-j,j}. \end{aligned}$$

Within $R_k$ we compare coefficients of $z^{k-j}\bar{z}^j$. For $k \neq 2j$, we find $V_{k-j,j}$ from the fact that

$$(k-2j)V_{k-j,j} = \text{coefficient of } z^{k-j}\bar{z}^j \text{ in } \sum_{\ell=2}^{k-1}(V_{\ell,\bar{z}}\bar{f}_{k-\ell+1} - V_{\ell,z}f_{k-\ell+1}). \tag{2.3}$$

Now

$$V_{\ell,z}f_{k-\ell+1} = \left(\sum_{m=0}^{\ell}(\ell-m)V_{\ell-m,m}z^{\ell-m-1}\bar{z}^m\right)\left(\sum_{n=0}^{k-\ell+1}A_{k-\ell+1,n}z^{k+1-\ell-n}\bar{z}^n\right).$$

We need to extract the coefficient of $z^{k-j}\bar{z}^j$ in this product: it is

$$C_{\ell kj} = \sum(\ell-m)V_{\ell-m,m}A_{k-\ell+1-n,n}$$

where the sum is over all $m, n$ with $m+n=j$ and such that $0 \leqslant m \leqslant \ell$, $0 \leqslant n \leqslant k+1-\ell$. Let $m = j-n$; then

$$C_{\ell kj} = \sum_n(\ell-j+n)V_{\ell-j+n,j-n}A_{k-\ell+1-n,n} \tag{2.4}$$

where the sum is now over $n$ in the range $\max(0, j-\ell) \leqslant n \leqslant \min(j, k+1-\ell)$. Similarly

$$V_{\ell,\bar{z}}\bar{f}_{k-\ell+1} = \left(\sum_{m=0}^{\ell}mV_{\ell-m,m}z^{\ell-m}\bar{z}^{m-1}\right)\left(\sum_{n=0}^{k+1-\ell}\bar{A}_{k+1-\ell-n,n}\bar{z}^{k+1-\ell-n}z^n\right).$$

The coefficient of $z^{k-j}\bar{z}^j$ in this product is

$$D_{\ell kj} = \sum mV_{\ell-m,m}\bar{A}_{k+1-\ell-n,n}$$

where the sum is over all $m, n$ with $n+\ell-m = k-j$ and $0 \leqslant m \leqslant \ell$, $0 \leqslant n \leqslant k+1-\ell$. Equivalently

$$D_{\ell kj} = \sum_n(n+\ell+j-k)V_{k-j-n,n+\ell-k+j}\bar{A}_{k+1-\ell-n,n} \tag{2.5}$$

where

$$\max(0, k-j-\ell) \leqslant n \leqslant \min(k-j, k+1-\ell).$$

In terms of $m$

$$D_{\ell kj} = \sum mV_{\ell-m,m}\bar{A}_{j+1-m,k-j+m-\ell} \tag{2.6}$$

with $\max(0, j+\ell-k) \leqslant m \leqslant \min(\ell, j+1)$. Note that $\ell \geqslant 2$ in the summation. Therefore

$$(k-2j)V_{k-j,j} = \sum_{\ell=2}^{k-1}(D_{\ell kj} - C_{\ell kj}) \tag{2.7}$$

where $C_{\ell kj}$ is given by (2.4) and $D_{\ell kj}$ by (2.5) or (2.6).

When $k = 2j$, this formula does not determine $V_{j,j}$. By definition, $V_{1,1} = 1$ and we have complete freedom in the choice of $V_{j,j}$ for $j > 1$; we set $V_{j,j} = 0$. At the same time we have

$$\eta_{2j} = \mathrm{i}\left(\sum_{\ell=2}^{k-1}(D_{\ell kj} - C_{\ell kj})\right) \tag{2.8}$$

with $k = 2j$ in the summation.

The focal values are given explicitly by (2.8). The procedure is iterative; the expression for $V_{k-j,j}$ given by (2.7) involves the $V_{i-j,j}$ with $i < k$. The corresponding algorithm is structurally simpler than that described in [6]. An implementation of this algorithm in REDUCE, called COMETA, was used to compute the focal values required in this paper. We shall demonstrate that, not only is it conceptually simpler, but that in certain circumstances it is also more efficient computationally.

## 3. Some useful results

Writing (1.3) in complex form leads to the algorithm described in section 2 and also to some useful theoretical results. We can establish that, in general, there are fewer terms in the focal values produced using COMETA, than by using FINDETA, for the same system. We also give results on the structure of the Liapunov function and the focal values. Finally, in this section we present the complex form of the two classical criteria for a critical point to be a centre.

We define a weight function $\mu$ on the set of monomials in the coefficients $A_{ij}$ and their complex conjugates as follows:

(i) $\mu(A_{ij}) = i - j - 1$;
(ii) $\mu(\bar{X}) = -\mu(X)$;
(iii) $\mu(kX) = \mu(X)$ if $k$ is a complex constant;
(iv) $\mu(XY) = \mu(X) + \mu(Y)$.

For a polynomial $p$ all of whose terms are of the same weight $\mu_0$, we define $\mu(p)$ to be $\mu_0$. We show that the focal values have weight zero.

**Lemma 1.** $\mu(V_{ij}) = i - j$.

**Proof.** The proof is by induction. Suppose that the result holds for $i + j < k$. From (2.5), each summand in $D_{\ell kj}$ is of weight $(k - j - n - (n + \ell - k + j)) - (k + 1 - \ell - n - n - 1) = k - 2j$. Similarly each summand in $C_{\ell kj}$ is of weight $(\ell - j + n - (j - n)) + (k - \ell + 1 - n - n - 1) = k - 2j$. The result follows. □

**Theorem 2.** *For all $j$, $\mu(\eta_{2j}) = 0$.*

**Proof.** When $k = 2j$, each term in $C_{\ell kj}$ and $D_{\ell kj}$ has weight zero. □

After the focal values have been calculated, the next step is to reduce each modulo the previous ones to obtain the Liapunov quantities. This is often demanding in terms of computing resources, and anything that simplifies the expressions obtained is useful. The focal values obtained by using FINDETA do not satisfy theorem 2. It is therefore apparent that the focal values given by (2.8) are already simpler in this sense.

The following lemma gives information on the form of the focal values and Liapunov function.

**Lemma 3.**

*(i) For all $k$ and $j$, $\bar{V}_{kj} = V_{jk}$.*
*(ii) For all $j$, $\eta_{2j}$ is real.*
*(iii) The $V_{kj}$ and $\eta_{2j}$ are polynomials in the $A_{mn}$ and $\bar{A}_{mn}$ with real coefficients.*

**Proof.**

(i) When $k = j$ the result is obvious. When $k \neq j$ we have

$$(k - j)V_{kj} = \sum_{\ell=2}^{k+j-1} (D_{\ell,k+j,j} - C_{\ell,k+j,j}).$$

Now

$$C_{\ell,k+j,k} = \sum_{n} (\ell - k + n)V_{\ell-k+n,k-n} A_{k+j-\ell+1-n,n}.$$

If we suppose that $V_{pq} = \bar{V}_{qp}$ for $p + q < k$, we have

$$\bar{C}_{\ell,k+j,k} = \sum_n (\ell - k + n) V_{k-n,\ell-k+n} \bar{A}_{k+j-\ell+1-n,n} = D_{\ell,k+j,j}.$$

It follows that $(k - j)\bar{V}_{kj} = -(j - k)V_{jk}$, whence $V_{kj} = \bar{V}_{jk}$.

(ii) Using the above in (2.8) we have

$$\eta_{2j} = \mathrm{i} \sum_{\ell=2}^{2j-1} (D_{\ell,2j,j} - C_{\ell,2j,j}) = \mathrm{i} \sum_{\ell=2}^{2j-1} (\bar{C}_{\ell,2j,j} - \bar{D}_{\ell,2j,j}) = \bar{\eta}_{2j}.$$

(iii) Again this follows by induction from the formulae (2.7) and (2.8).

□

It is not always as helpful as one might expect to incorporate these results in the computer algorithm. Computing $V_{ij}$ and using its complex conjugate when $V_{ji}$ is required is generally slower than computing the $V_{ji}$ directly. This is due, in part, to the lack of an in-built facility in REDUCE for handling a data structure of type 'complex'. Some saving in space can be achieved and this may be significant for some problems. Similarly the focal values could be obtained from the imaginary part of the appropriate $C$ (or $D$) but the expression for $C$ alone may contain more terms than $D - C$.

Writing system (1.1) in complex form has some advantages when proving the sufficiency of the conditions for integrability. The two simplest criteria are that a critical point of focus type is a centre if the system is Hamiltonian (that is, $P_x + Q_y = 0$) or the system is symmetric in a line through the origin. The corresponding results for the system

$$\mathrm{i}\dot{z} = z + \sum_{k=2}^{N} \sum_{\ell=0}^{k} A_{k-\ell,\ell} z^{k-\ell} \bar{z}^{\ell} \tag{3.1}$$

are as follows.

**Lemma 4.** *The origin is a centre for (3.1) if*

$$(k - \ell) A_{k-\ell,\ell} = (\ell + 1) \bar{A}_{\ell+1,k-1-\ell}$$

*for all* $k, \ell$ *with* $2 \leqslant k \leqslant N$ *and* $0 \leqslant \ell < \frac{1}{2}k$.

**Proof.** Let the nonlinearity on the right-hand side of (3.1) be $f = u + \mathrm{i}v$. Then we have

$$\dot{x} = y + v \qquad \dot{y} = -x - u.$$

So the divergence of the vector field is $v_x - u_y$, which is $-\mathrm{i} f_z + \mathrm{i} \bar{f}_{\bar{z}}$. But

$$\begin{aligned} f_z - \bar{f}_{\bar{z}} &= \sum_{k=2}^{N} \sum_{\ell=0}^{k-1} (k - \ell)(A_{k-\ell,\ell} z^{k-\ell-1} \bar{z}^{\ell} - \bar{A}_{k-\ell,\ell} \bar{z}^{k-\ell-1} z^{\ell}) \\ &= \sum_{k=2}^{N} \sum_{\ell=0}^{k-1} ((k - \ell) A_{k-\ell,\ell} - (\ell + 1) \bar{A}_{\ell+1,k-1-\ell}) z^{k-\ell-1} \bar{z}^{\ell}. \end{aligned}$$

Hence the system is Hamiltonian if and only if

$$\alpha_{k,\ell} = (k - \ell) A_{k-\ell,\ell} - (\ell + 1) \bar{A}_{\ell+1,k-1-\ell} = 0$$

for all $k, \ell$ with $2 \leqslant k \leqslant N$ and $0 \leqslant \ell \leqslant k - 1$. But $\bar{\alpha}_{k,\ell} = -\alpha_{k,k-1-\ell}$; the result follows. □

**Lemma 5.** *The origin is a centre for (3.1) if* $A_{21}$ *is real and there is* $\theta$ *such that* $A_{kj} \mathrm{e}^{\mathrm{i}(k-j-1)\theta}$ *are all real.*

**Proof.** Under the transformation $z \mapsto z\mathrm{e}^{\mathrm{i}\theta}$, the equation becomes

$$\mathrm{i}\dot{z} = z + \sum A_{kj}\mathrm{e}^{\mathrm{i}(k-j-1)\theta} z^k \bar{z}^j. \tag{3.2}$$

If the original system is symmetric in a line through the origin there is $\theta$ such that (3.2) is symmetric in the $x$-axis. The condition for this is $A_{kj}\mathrm{e}^{\mathrm{i}(k-j-1)\theta} \in \mathbb{R}$. □

The symmetry of a system in a line is not obvious when the real form is used except when the line is one of the axes. In complex form it is immediate. The following result follows directly from lemma 5 with $\theta = 0$.

**Corollary 6.** *Suppose that $A_{kj} \in \mathbb{R}$ for all $k, j$. Then the origin is a centre for (3.1).*

In contrast to corollary 6, the origin is not a centre when the $A_{kj}$ are pure imaginary. In the next section we resolve the centre focus question in this case with $A_{11} = 0$.

## 4. Examples

We first illustrate the use of COMETA by applying it in the case of the system

$$\mathrm{i}\dot{z} = z + A_{30}z^3 + A_{21}z^2\bar{z} + A_{12}z\bar{z}^2 + A_{03}\bar{z}^3. \tag{4.1}$$

The centre conditions for (4.1) are well known, see [12] for example. We refer to this system here for two reasons. First, it serves as a test of the algorithm and, secondly, we are able to compare the use of the complex form with other methods of computing focal values and deriving integrability conditions. We can readily find the necessary centre conditions using COMETA as shown below. In contrast the calculations for the real form are of comparable difficulty only after a linear change of coordinates; this transformation then has to be inverted to obtain the conditions for the original system. So for this system it is clearly advantageous to use the complex form to find the necessary centre conditions. We also have that the sufficiency of one of the centre conditions is proved more readily in the complex form. The corresponding case in the real form required the search for invariant curves from which a Dulac function was constructed [12].

We use COMETA to calculate the focal values, $\eta_{2j}$, for (4.1) and hence obtain the corresponding Liapunov quantities, $L(i)$. Here $L(i)$ is obtained from $\eta_{2i+2}$ modulo $\eta_{2j}$ $(2 \leqslant j \leqslant i)$ with non-zero multiplicative factors removed. We find that $L(1) = A_{21} - \bar{A}_{21}$ and with $\bar{A}_{21} = A_{21}$ then

$$L(2) = -A_{30}A_{12} + \bar{A}_{30}\bar{A}_{12}.$$

Assume for the time being that $A_{12} \neq 0$ and let $\bar{A}_{30} = A_{30}A_{12}/\bar{A}_{12}$ then

$$L(3) = (3A_{30} - \bar{A}_{12})(A_{12}^2\bar{A}_{03} - \bar{A}_{12}^2A_{03})(A_{30} + 3\bar{A}_{12})$$
$$L(4) = A_{21}(3A_{30} - \bar{A}_{12})(A_{12}^2\bar{A}_{03} - \bar{A}_{12}^2A_{03})(A_{30} + 7\bar{A}_{12})$$
$$L(5) = (3A_{30} - \bar{A}_{12})(A_{12}^2\bar{A}_{03} - \bar{A}_{12}^2A_{03})F$$

where $F$ is a polynomial in $A_{30}, A_{21}, A_{12}, \bar{A}_{12}, A_{03}, \bar{A}_{03}$. Hence it is possible that the origin is a centre when $A_{12} \neq 0$ if either of the conditions

$$\mathrm{Im}\,(A_{21}) = 0 \qquad 3A_{30} - \bar{A}_{12} = 0$$

or

$$\mathrm{Im}\,(A_{21}) = \mathrm{Im}\,(A_{30}A_{12}) = \mathrm{Im}\,(A_{12}^2\bar{A}_{03}) = 0$$

holds. If neither of these holds then $L(3) = 0$ only if $A_{30} + 3\bar{A}_{12} = 0$. Then we have $L(4) = A_{21}$ and with $A_{21} = 0$, $L(5) = 4A_{12}\bar{A}_{12} - A_{03}\bar{A}_{03}$ and $L(6) = 0$. Another possible condition for the origin to be a centre when $A_{12} \neq 0$ is

$$A_{21} = \mathrm{Im}\,(A_{30}A_{12}) = 0 \qquad A_{30} + 3\bar{A}_{12} = 0 \qquad 4A_{12}\bar{A}_{12} - A_{03}\bar{A}_{03} = 0.$$

Returning to the case with $A_{12} = 0$ we have $L(3) = A_{30}^2 A_{03} - \bar{A}_{30}^2 A_{03}$ which is a factor of $L(4)$, $L(5)$ and $L(6)$. Hence another possible centre condition is

$$\mathrm{Im}\,(A_{21}) = A_{12} = \mathrm{Im}\,(A_{30}^2 A_{03}) = 0.$$

These four necessary conditions for the origin to be a centre can be combined to give the following result.

**Theorem 7.** *The origin is a centre for (4.1) if and only if one of the following holds:*

*(i)* $\mathrm{Im}\,(A_{21}) = 0$, $3A_{30} - \bar{A}_{12} = 0$;
*(ii)* $\mathrm{Im}\,(A_{21}) = \mathrm{Im}\,(A_{30}A_{12}) = \mathrm{Im}\,(A_{03}\bar{A}_{12}^2) = \mathrm{Im}\,(A_{30}^2 A_{03}) = 0$;
*(iii)* $A_{21} = 0$, $A_{30} + 3\bar{A}_{12} = 0$, $4A_{12}\bar{A}_{12} - A_{03}\bar{A}_{03} = 0$.

**Proof.** Necessity has already been confirmed. When (i) holds, the system is Hamiltonian and hence the origin is a centre by lemma 4. If (ii) is satisfied then by lemma 5 the system is symmetric in a line through the origin, again the origin is a centre. In [12] we proved the sufficiency of this condition by finding an invariant conic and constructing a Dulac function. In the complex form symmetry in a line is immediate. It was also shown in [12] that when (iii) holds, an invariant quartic, $S$, exists and $S^{-5/2}$ is a Dulac function. □

We saw earlier that if all the coefficients $A_{ij}$ are real then the origin is a centre for (1.3), and hence there are no limit cycles encircling the origin. The other extreme is when the $A_{ij}$ are pure imaginary. Calculation of the focal values and, in particular, their reduction is computationally very demanding in this case. However, such a system with $A_{11} = 0$ is tractable and we present the results below. The system is interesting in that although it has only six non-zero pure imaginary coefficients it can have six small amplitude limit cycles.

We consider

$$\mathrm{i}\dot{z} = z + A_{20}z^2 + A_{02}\bar{z}^2 + A_{30}z^3 + A_{21}z^2\bar{z} + A_{12}z\bar{z}^2 + A_{03}\bar{z}^3 \tag{4.2}$$

where $A_{20} = \mathrm{i}a_2$, $A_{02} = \mathrm{i}c_2$, $A_{30} = \mathrm{i}d_2$, $A_{21} = \mathrm{i}f_2$, $A_{12} = \mathrm{i}g_2$, $A_{03} = \mathrm{i}h_2$. We note that the origin is a centre for the quadratic part of (4.2). We use COMETA to calculate the focal values. We find that $L(1) = A_{21}$ and with $A_{21} = 0$, $L(2) = A_{02}A_{20}(A_{03} + A_{12} - A_{30})$.

First we let $A_{02} = 0$; then $L(3) = A_{03}(A_{12} + 3A_{30})(A_{30} - 3A_{12})$. When $A_{03} = 0$ we find $L(4) = A_{20}^2 A_{12}(A_{12} - A_{30})^2$ is a factor of $L(5)$ and $L(6)$, suggesting that the origin is a centre if

$$A_{21} = A_{02} = A_{03} = 0 \qquad A_{20}A_{12}(A_{12} - A_{30}) = 0. \tag{4.3}$$

When $A_{12} = -3A_{30}$, so that $L(3) = 0$, then $L(4) = A_{20}^2 A_{30}$ and $A_{20}$ is a factor of $L(5)$. The origin may be a centre if

$$A_{21} = A_{02} = A_{20} = 0 \qquad 3A_{30} + A_{12} = 0. \tag{4.4}$$

All other possible conditions for a centre with $A_{30} = 0$ are covered by (4.3).

Also $L(3) = 0$ when $A_{30} = 3A_{12}$; then $L(4) = A_{20}^2 A_{12}(A_{03} + 4A_{12})(19A_{03} - 10A_{12})$. If $A_{20} = 0$ then $L(5) = A_{03}A_{12}(A_{03}^2 - 4A_{12}^2)$. The case with $A_{03} = 0$ is covered by (4.3), that with $A_{12} = 0$ by (4.4) and if

$$A_{21} = A_{02} = A_{20} = 0 \qquad A_{30} - 3A_{12} = 0 \qquad A_{03}^2 - 4A_{12}^2 = 0 \tag{4.5}$$

then the origin may be a centre. On the other hand, if $A_{20} \neq 0$ then all possible centre conditions are subcases of (4.3) and (4.4).

Another situation in which $L(2) = 0$ is when $A_{30} = A_{03} + A_{12}$. Then we find $L(3) = A_{03}(A_{03} - 2A_{12})(3A_{03} + 4A_{12})$ and $A_{03}$ is a factor of $L(4)$ and $L(5)$. The origin may be a centre if

$$A_{21} = A_{03} = 0 \qquad A_{30} = A_{12}. \tag{4.6}$$

The possible centre condition with $A_{12} - A_{30} = 0$ in (4.3) is covered by (4.6). With $A_{03} = 2A_{12}$ all possible centre conditions are already covered. When $3A_{03} + 4A_{12} = 0$, $A_{20}$ is a factor of $L(4)$, $L(5)$ and the origin may be a centre if

$$A_{21} = A_{20} = 0 \qquad 3A_{03} + 4A_{12} = 0 \qquad A_{03} + A_{12} - A_{30} = 0. \tag{4.7}$$

Again all other possibilities are covered by the conditions above.

Finally $L(2) = 0$ if $A_{20} = 0$ and then $L(3) = A_{03}(3A_{30} + A_{12})(A_{30} - 3A_{12})$. As $3A_{30} + A_{12}$ is also a factor of $L(4)$, $L(5)$, $L(6)$ the origin may be a centre if

$$A_{21} = A_{20} = 0 \qquad 3A_{30} + A_{12} = 0 \tag{4.8}$$

which covers (4.4) and (4.7). When $A_{03} = 0$ then

$$L(4) = A_{02}^2(A_{12} - A_{30})(A_{12} + 3A_{30})(2A_{12} - A_{30})$$

and

$$L(5) = A_{02}^4(A_{12} - A_{30})(A_{12} + 3A_{30})(25A_{12} + 19A_{30}).$$

All situations which may lead to a centre are subcases of those conditions already found. When $A_{30} - 3A_{12} = 0$ no new possible centre conditions arise; the details will be given later when we consider the bifurcation of limit cycles.

**Theorem 8.** *The origin is a centre for (4.2) if and only if one of the following holds:*

*(i)* $A_{20} = A_{21} = 0$, $3A_{30} + A_{12} = 0$;
*(ii)* $A_{20} = A_{02} = A_{21} = A_{03} = 0$;
*(iii)* $A_{02} = A_{21} = A_{12} = A_{03} = 0$;
*(iv)* $A_{21} = A_{03} = 0$, $A_{30} = A_{12}$;
*(v)* $A_{20} = A_{02} = A_{21} = 0$, $A_{30} - 3A_{12} = 0$, $A_{03}^2 - 4A_{12}^2 = 0$.

**Proof.** Necessity has been proved. When (i) holds the system is Hamiltonian and hence the origin is a centre. The system is symmetric in a line through the origin when (ii) holds and thus the origin is a centre. In fact in this instance the system is of the form (4.1) and, as $A_{12}A_{30}$ is real, condition (ii) of theorem 7 holds.

To prove the sufficiency of conditions (iii)–(v) we revert to the real form of the equations and use the technique described in [12] to construct Dulac functions from invariant curves. When (iii) holds we find invariant lines can be used to give a Dulac function $B = \prod_{i=1}^{4} L_i^{\alpha_i}$ with

$$L_i = m_i x - a_2 y + \frac{d_2}{m_i} y + 1$$

and

$$\alpha_i = \frac{-(2a_2 m_i - 3d_2)}{(a_2 m_i - 2d_2)}$$

where the $m_i$ are distinct non-zero roots of $m^4 + a_2^2 m^2 - 2a_2 d_2 m + d_2^2 = 0$. If double roots arise, or if any of the $m_i$ is zero or equal to $2d_2/a_2$, then the system reduces to a quadratic and as has already been noted the origin is a centre for the quadratic part of (4.2).

When (iv) holds the system falls into the class covered by condition (3) of lemma 4.2 in [12]. Again there are invariant lines and here we require three of them to form the Dulac function $B = \prod_{i=1}^{3} L_i^{\alpha_i}$ where

$$L_i = m_i x - (a_2 + c_2)y + \frac{2g_2}{m_i} y + 1$$
$$\alpha_i = 2m_i(c_2 m_j m_k - 2g_2(m_j + m_k))/g_2(m_i(m_j + m_k) - m_i^2 - m_j m_k)$$

with $j = i + 1(\mathrm{mod}\,3)$, $k = i + 2(\mathrm{mod}\,3)$. The $m_i, m_j, m_k$ are three distinct non-zero roots of $m^4 + m^2(a_2 - 3c_2)(a_2 + c_2) + 8c_2 g_2 m - 4g_2^2 = 0$. When there are not three distinct non-zero roots to this equation the system is quadratic and as before the origin is a centre.

The system is of the form (4.1) when (v) holds and is covered by condition (iii) of theorem 7. The Dulac function in this case involves an invariant quartic; we find $B = S^{-5/2}$, where $S = 16g_2^2 y^4 - 8g_2 xy + 1$ when $A_{03} = 2A_{12}$ and where $S = 16g_2^2 x^4 - 8g_2 xy + 1$ when $A_{03} = -2A_{12}$. □

Having established the conditions under which the origin is a centre for (4.2) we can determine the maximum order of the origin as a fine focus. Hence we can find the maximum number of small amplitude limit cycles that can be bifurcated from the origin.

For a fine focus of order greater than one we must have $A_{21} = 0$ and if $A_{20} = 0$ also the origin is of order at least two. We then have $L(3) = A_{03}(3A_{30} + A_{12})(A_{30} - 3A_{12})$. If $3A_{30} + A_{12} = 0$ then the origin is a centre and if $A_{03} = 0$ the origin is a centre or a fine focus of order at most five. If $A_{30} - 3A_{12} = 0$ the order is at least four. Then $L(4) = A_{02}^2 A_{12}(A_{03} - 2A_{12})(7A_{03} - 4A_{12})$ and if $A_{03} = 4A_{12}/7$ the origin is a fine focus of order at least five. Now $L(5) = -A_{12}^3(245A_{02}^4 + 144A_{12}^2)$ and for a fine focus of order six we let $144A_{12}^2 = -245A_{02}^4$; then $L(6) = -A_{02}^{10} A_{12}$. If $L(6) = 0$ the origin is a centre, hence the order of the origin as a fine focus is at most six.

We proceed to show that six limit cycles can be bifurcated in this instance. We write the system in real form:

$$\begin{aligned}\dot{x} &= \lambda x + y + (a_2 + c_2)x^2 - (a_2 + c_2)y^2 + (d_2 + f_2 + g_2 + h_2)x^3 \\ &\quad -(3d_2 - f_2 - g_2 + 3h_2)xy^2 \\ \dot{y} &= -x + \lambda y + 2(a_2 - c_2)xy + (3d_2 + f_2 - g_2 - 3h_2)x^2 y - (d_2 + f_2 + g_2 - h_2)y^3.\end{aligned} \tag{4.9}$$

**Theorem 9.** *The origin is a fine focus of maximum order six for (4.9). It is of order six if*

$$\lambda = a_2 = f_2 = 0 \qquad d_2 = 3g_2 \qquad h_2 = 4g_2/7 \qquad g_2^2 = 245c_2^4/144 \qquad \textit{and} \qquad c_2 \neq 0.$$

To obtain six limit cycles we perturb the coefficients, one at a time, so that the stability of the origin is reversed at each perturbation and a limit cycle bifurcates. First we summarize the Liapunov quantities in terms of the real coefficients. Here the sign of the Liapunov quantity is important; we define $\mathcal{L}(i)$ to be the reduced focal value $\eta_{2i+2}$ with strictly positive multiplicative factors removed.

$$\mathcal{L}(0) = \lambda \qquad \mathcal{L}(1) = f_2$$
$$\mathcal{L}(2) = a_2 c_2(h_2 + g_2 - d_2)$$
$$\mathcal{L}(3) = h_2(3g_2 - d_2)(g_2 + 3d_2)$$
$$\mathcal{L}(4) = c_2^2 g_2(h_2 - 2g_2)(7h_2 - 4g_2)$$
$$\mathcal{L}(5) = g_2^3(245c_2^4 - 144g_2^2)$$
$$\mathcal{L}(6) = c_2^{10} g_2.$$

When the relationships given in theorem 9 hold, $\mathcal{L}(6)$ is non-zero and all the other Liapunov quantities are zero. We aim to show that six limit cycles can be bifurcated; to simplify the procedure we assume that $g_2 > 0$. A reversal of the stability of the origin is achieved by increasing $g_2$ so that $\mathcal{L}(5)\mathcal{L}(6) < 0$. At the same time $d_2$ and $h_2$ are adjusted to ensure $\mathcal{L}(3)$ and $\mathcal{L}(4)$ remain equal to zero. The first limit cycle bifurcates. To obtain the second limit cycle $h_2$ is decreased, $\mathcal{L}(4)$ and $\mathcal{L}(5)$ are now of opposite sign, and the stability of the origin is again reversed. Increasing $d_2$ will make $\mathcal{L}(3)\mathcal{L}(4) < 0$ and a third limit cycle bifurcates. For the fourth limit cycle, $a_2$ is perturbed so that $a_2c_2 < 0$ and decreasing $f_2$ produces a fifth limit cycle. The sixth and final limit cycle comes by increasing $\lambda$.

## 5. The Kukles system

Finally we consider another particular case of (3.1), namely that in which $A_{ij} = \bar{A}_{ji}$. The real form of such equations is

$$\dot{x} = y \qquad \dot{y} = -x + \sum_{i+j\geqslant 2} a_{ij}x^i y^j. \tag{5.1}$$

In previous papers [7, 8] we investigated cubic systems of this form, and showed that the centre conditions proposed by Kukles [5] are incomplete. We saw in [8] that the search for necessary conditions for a centre can be extremely demanding of computing resources and considerable ingenuity is often required in the reduction of the focal values. We gave all integrability conditions which are *persistent* in a sense which we defined and we conjectured that there are no others: we now confirm that this is so.

A centre is *persistent* if it is preserved by some perturbation of the coefficients. This, of course, must be relative to a specific class of perturbations, and this has to take account of the fact that every condition is preserved under scaling of the coefficients. In the case of the systems

$$\dot{x} = y \qquad \dot{y} = -x + \sum_{i+j=2}^{3} a_{ij}x^i y^j \tag{5.2}$$

we gave all centre conditions when $a_{11} = 0$ in [7]; we scaled the coefficients by $a_{11}$ in [8] and found all persistent centres of (5.2).

**Theorem 10.** *The system (5.1) with $a_{11} \neq 0$ has no non-persistent centres.*

**Proof.** A non-persistent centre of (5.1) with $a_{11} = 1$ is given by relations of the form $A_{ij} = \lambda_{ij}$, where each $\lambda_{ij}$ is a complex number which is not a function of the $A_{ij}$. For the unscaled system these relations become $A_{ij} = \lambda_{ij}a_{11}^{i+j-1}$. The complex form of (5.1) is invariant under the transformation $(z, t) \mapsto (\bar{z}, -t)$. The centre conditions are therefore invariant under conjugation. Therefore the $\lambda_{ij}$ are real. It follows that all the coefficients $A_{ij}$ are real. The origin is necessarily a centre (corollary 6), but it is persistent because it is preserved by every perturbation in which the $A_{ij}$ remain real. $\square$

**Corollary 11.** *The origin is a centre for the Kukles system (5.2) if and only if one of the conditions given in theorem 3.1 of [8] holds.*

**Remark.** In theorem 10 we have supposed that $a_{11} \neq 0$; a similar result holds if any specific $a_{ij}$ is used to scale (5.1).

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